

# EDGE NUMBER OF KNOTS AND LINKS

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ABSTRACT. We introduce a new numerical invariant of knots and links made from the partitioned diagrams. It measures the complexity of knots and links.

## 1. WHY DOES NOT A KNOT COME LOOSE?

There are a lot of answers to a question “Why does not a knot come loose?”. The next example gives one of the answers.

In Figure 1, the trefoil knot diagram is partitioned by 3 vertices into 3 edges  $e_1, e_2$  and  $e_3$ . The edge  $e_1$  is over  $e_2$ ,  $e_2$  is over  $e_3$  and  $e_3$  is over  $e_1$ . This relation is known to be a three-way deadlock. It seems that this is one of the factors that why a knot cannot come loose.

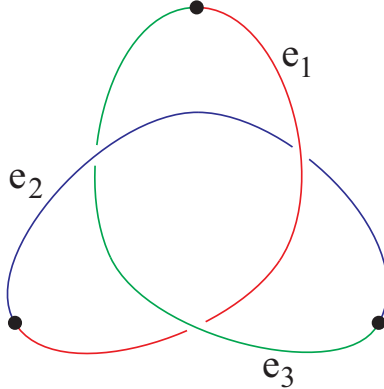


FIGURE 1. a 3-edge presentation

## 2. INTRODUCTION TO THE DEFINITION OF EDGE NUMBER

Throughout this paper we work in the piecewise linear category. We shall study knots and links in the three-dimensional Euclidean space  $\mathbb{R}^3$ . For the standard definitions and results of knots and links, we refer to [1], [2], [3], [5], [6], [7], [9] and [12].

Let  $K$  be a knot or link and  $D$  be a diagram of  $K$ . We partition  $D$  by  $n$  vertices into  $n$  edges  $e_1, \dots, e_n$ . We assume that each component of  $K$  has at least one vertex. We call such a diagram  $D$  an  $n$ -partitioned diagram.

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**Definition 2.1.** An  $n$ -partitioned diagram  $D$  is an  $n$ -cycle presentation of  $K$  if

- (1) Each edge of  $D$  has no self-crossing.
- (2) For any pair of two edges  $e_i$  and  $e_j$ , exactly one of the following holds.
  - (a)  $e_i$  is over  $e_j$  at every crossing of  $e_i$  and  $e_j$ .
  - (b)  $e_i$  is under  $e_j$  at every crossing of  $e_i$  and  $e_j$ .
  - (c) There is no crossing of  $e_i$  and  $e_j$ .

We define an *edge number*  $e(K)$  of  $K$  as the minimum number of  $n$  where  $n$  is taken over all  $n$ -cycle presentation of  $K$ .

We can make a directed graph  $G(D)$  from an  $n$ -cycle presentation  $D$  as follows.

- (1) For each edge  $e_i$ , we assign a vertex  $v_i$  to  $G(D)$ .
- (2) For two edges  $e_i$  and  $e_j$  which have at least one crossing, if  $e_i$  is over (resp. under)  $e_j$ , then we assign an oriented edge from  $e_i$  to  $e_j$  (resp. from  $e_j$  to  $e_i$ ) to  $G(D)$ .

Figure 2 shows a digraph  $G(D)$  obtained from a 3-edge presentation  $D$  in Figure 1.

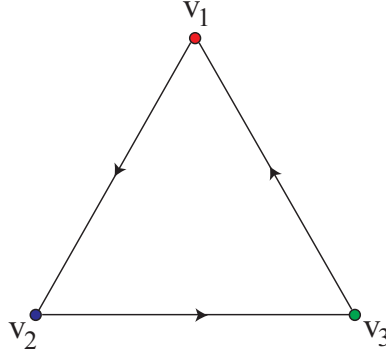


FIGURE 2. the digraph of a 3-edge presentation in Figure 1

*Remark 2.2.* If we exclude the condition (2) in the definition of an  $n$ -cycle presentation, then any knot has a 2-cycle presentation. Figure 3 will help you to show this.

### 3. RESULTS AND CONJECTURE

**Proposition 3.1.** A knot  $K$  is non-trivial if and only if  $e(K) \geq 3$ .

**Proposition 3.2.** Let  $K$  be a knot with  $e(K) = 3$  and  $D$  a 3-cycle presentation of  $K$ . Then  $G(D)$  is an oriented 3-cycle.

**Theorem 3.3.** For any minimal  $n$ -cycle presentation  $D$ ,

- (1)  $G(D)$  is connected.
- (2)  $G(D)$  is not a path.

**Conjecture 3.4.** For a non-trivial knot  $K$  and any  $n$ -cycle presentation  $D$  of  $K$ ,  $G(D)$  contains at least one oriented cycle.

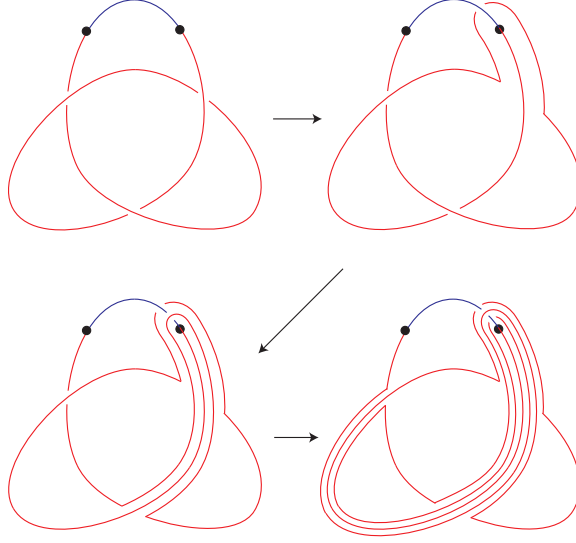


FIGURE 3. 2-cycle presentation without the condition (2)

**Proposition 3.5.** *For any  $n \geq 3$  ( $n \equiv 0, 1, 2, 3, 4 \pmod{6}$ ), there exists a knot such that  $c(K) = n$  and  $e(K) = 3$ .*

**Conjecture 3.6.**  $b(K) \leq e(K)$ .

*Remark 3.7.*  $e(K) \leq 2b(K)$ .

**Problem 3.8.** Find a knot  $K$  with  $e(K) = 4$ . We expect that  $5_1$  is a candidate.

#### 4. PROOFS

*Proof.* (of Proposition 3.1)

( $\Rightarrow$ ) Suppose that  $e(K) = 1$  and let  $D$  be a 1-cycle presentation of  $K$ . Then by the definition (1) of an  $n$ -cycle presentation,  $D$  has no crossing. Hence  $K$  is trivial. Next suppose that  $e(K) = 2$  and let  $D$  be a 2-cycle presentation of  $K$ .  $D$  has two edges  $e_1$  and  $e_2$  and without loss of generality we may assume that  $e_1$  is over  $e_2$  at every crossings of  $e_1$  and  $e_2$ . Then by the definition (1) of an  $n$ -cycle presentation,  $D$  is a 1-bridge presentation with an over-bridge  $e_1$  and an under-bridge  $e_2$ . Hence  $K$  is trivial.

( $\Leftarrow$ ) Suppose that  $K$  is trivial. Then  $K$  has a 1-cycle presentation. Hence  $e(K) = 1$ .  $\square$

*Proof.* (of Proposition 3.2) Let  $D$  be a 3-cycle presentation of  $K$  with three edges  $e_1, e_2$  and  $e_3$ . Suppose that  $G(D)$  does not form an oriented 3-cycle. Without loss of generality, we may assume that  $e_1$  is over  $e_2$  at every crossings of  $e_1$  and  $e_2$ ,  $e_2$  is over  $e_3$  at every crossings of  $e_2$  and  $e_3$ , and  $e_1$  is over  $e_3$  at every crossings of  $e_1$  and  $e_3$ . Then  $D$  is a descending diagram by specifying an orientation in the order  $e_1, e_2, e_3$ . Therefore  $K$  is trivial and this contradicts  $e(K) = 3$  and Proposition 3.1. Hence  $G(D)$  forms an oriented 3-cycle.  $\square$

**Lemma 4.1.** *Let  $K$  be a knot with  $e(K) = n$  and  $D$  an  $n$ -cycle presentation of  $K$ . Then*

- (1) *For any successive vertices  $v_i, v_{i+1}$  of  $G(D)$ , both of them can not be a source nor sink in  $G(D) - v_i v_{i+1}$ .*
- (2) *For any successive non-adjacent vertices  $v_i, v_{i+1}$  of  $G(D)$ , there exists a vertex  $v_k$  in  $N(v_i) \cap N(v_{i+1})$  such that  $v_i v_k v_{i+1}$  is an oriented path,*

where  $N(v)$  denotes the set of vertices which is adjacent to  $v$ .

*Proof.* (of Lemma 4.1) Let  $K$  be a knot with  $e(K) = n$  and  $D$  an  $n$ -cycle presentation of  $K$ .

(1) Suppose without loss of generality that there exist successive vertices  $v_i, v_{i+1}$  of  $G(D)$  such that both of them are source in  $G(D) - v_i v_{i+1}$ . Since the subarc  $e_i \cup e_{i+1}$  is over all other edges  $e_j$  ( $j \neq i, i+1$ ), there is an isotopy of  $D$  such that  $e_i \cup e_{i+1}$  has no self-crossing. Then we can regard the subarc  $e_i \cup e_{i+1}$  as a single edge  $e'_i$ , and the new edge  $e'_i$  satisfies the definition (1) and (2) of an  $n$ -cycle presentation. See Figure 4. Hence we obtain an  $(n-1)$ -cycle presentation of  $K$  and this contradicts  $e(K) = n$ .

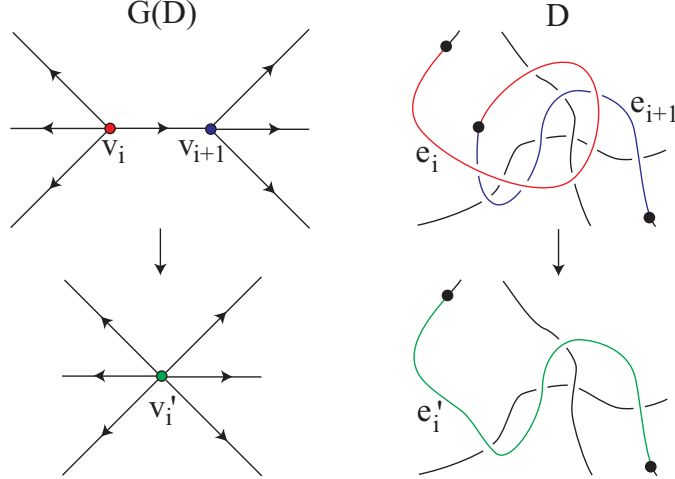
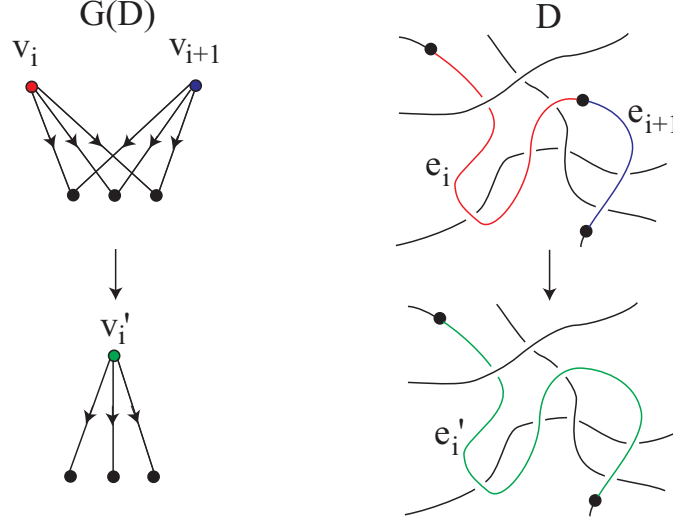


FIGURE 4. reducing  $e_i \cup e_{i+1}$

(2) Suppose without loss of generality that there exist successive non-adjacent vertices  $v_i, v_{i+1}$  such that for any vertex  $v_k$  in  $N(v_i) \cap N(v_{i+1})$ , the edge  $v_i v_k$  (resp.  $v_{i+1} v_k$ ) has an orientation from  $v_i$  to  $v_k$  (resp. from  $v_{i+1}$  to  $v_k$ ). We regard the subarc  $e_i \cup e_{i+1}$  as a single edge  $e'_i$ . Then  $e'_i$  satisfies the definition (1) and (2) of an  $n$ -cycle presentation. See Figure 5. Hence we obtain an  $(n-1)$ -cycle presentation of  $K$  and this contradicts  $e(K) = n$ .  $\square$

*Remark 4.2.* By Lemma 4.1 (2), for any successive vertices  $v_i, v_{i+1}$ ,  $d(v_i, v_{i+1}) \leq 2$ , where  $d(v, v')$  denotes the distance between  $v$  and  $v'$  in the graph.

*Proof.* (of Theorem 3.3) Let  $D$  be a minimal  $n$ -cycle presentation.

FIGURE 5. reducing  $e_i \cup e_{i+1}$ 

(1) Suppose that  $G(D)$  is disconnected. Then there exist successive vertices  $v_i, v_{i+1}$  such that  $v_i$  is not connected to  $v_{i+1}$  in  $G(D)$ . This contradicts Remark 4.2.

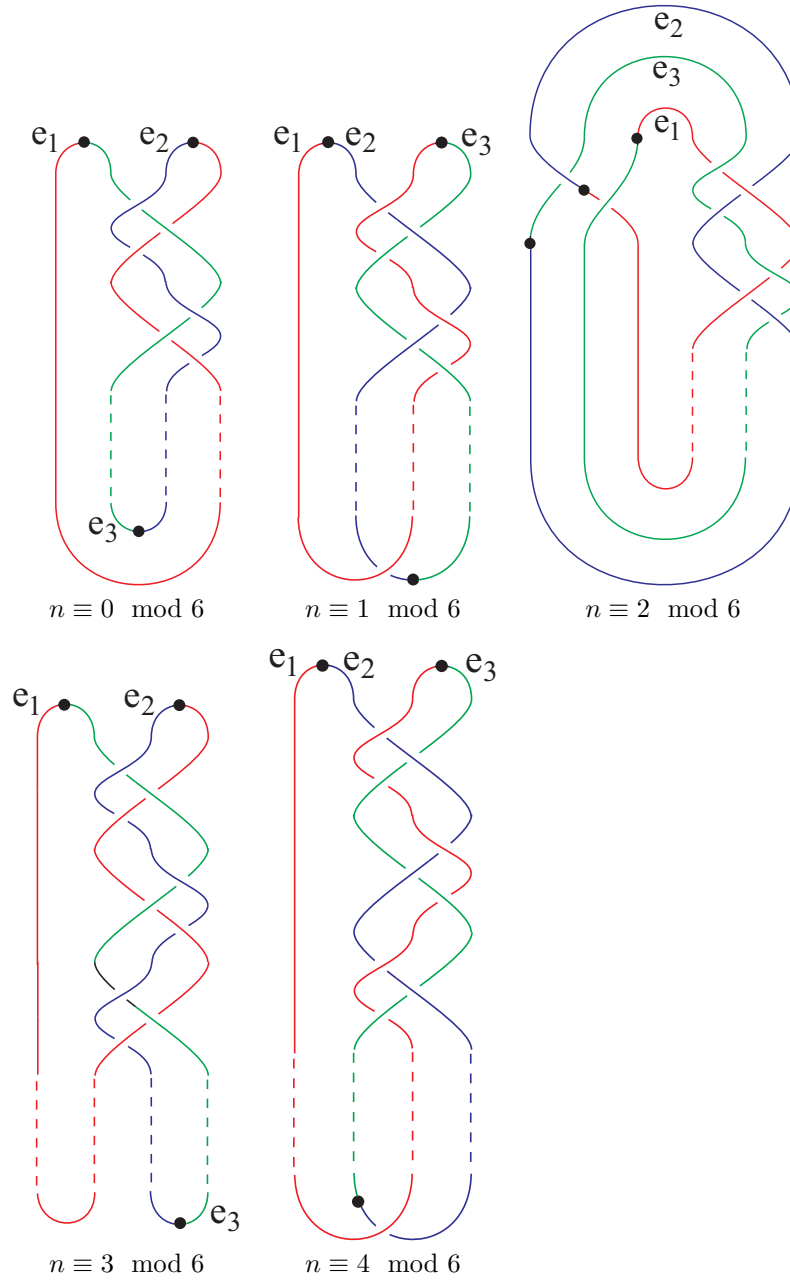
(2) Suppose that  $G(D)$  is a path. Then by Remark 4.2,  $G(D)$  has length two. This contradicts Proposition 3.2.  $\square$

*Proof.* (of Proposition 3.5) The demanded knots are displayed in Figure 6. By [4], [8] and [13], these diagrams are minimal crossing since they are alternating diagrams.  $\square$

*"Proof" of Conjecture 3.6.* Let  $K$  be a knot with  $e(K) = n$  and  $D$  an  $n$ -cycle presentation of  $K$  on the 2-sphere  $S^2$  dividing the 3-sphere  $S^3$  into two 3-balls  $B_+$  and  $B_-$ . We push all subarcs of  $D$ , which are regular neighbourhoods of vertices, into  $B_-$ , and pull the rest of  $D$  into  $B_+$ . Then  $K$  intersects  $S^2$  in  $2n$ -points, and  $K \cap B_{\pm}$  consists of properly embedded  $n$ -arcs in  $B_{\pm}$ . Conjecture 3.6 will be proved if we can show that the  $n$ -string tangle  $(B_+, K \cap B_+)$  is trivial.

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FIGURE 6. knots with  $e(K) = 3$ 

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